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The Phase Space Formalism for Quantum Mechanics and C* Axioms

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Abstract On any quantum mechanical Hilbert space, the phase space localization operators form a set of operators that are both physically motivated and form the groundwork for a C* algebra. This set is shown to be informationally complete in the original Hilbert space. We also revisit the relation between having a complete set of eigenvectors, commutability and compatibility.

Keywords C* algebra · Quantum mechanics · Phase space · Informational completeness

1 Introduction

We have discussed the phase space formulation of quantum mechanics at these conferences, and most recently have abstracted an algebra of operators (the phase space localization operators) from it that is an M.V. algebra and a Heyting algebra. Along the way, we discussed the informational completeness of this set of operators in certain Hilbert space representations, such as the usual massive, spinless particle representation. In this paper, we will assert that *every* phase space representation of quantum mechanics has this informational completeness for the phase space localization operators.

In Sect. 2, we review the axioms for a C* algebra by following the program of G.G. Emch in which a C* algebra for a physical system is treated. In Sect. 3, we describe informational completeness in any Hilbert space. In the next section, we briefly obtain the phase space, Γ, and the Lie group that is associated with Γ. We shall also obtain the set of square integrable functions over Γ, the representation we obtain on this Hilbert space, and the operators of multiplication by functions of the coordinates. In Sect. 5, we intertwine the irreducible Hilbert space(s) of ordinary quantum mechanics with $L^2(\Gamma)$ and obtain the pull-back of the operators of multiplication by functions of the coordinates in phase space. In Sect. 6, we look at the spectral properties of these pulled-back operators, and revisit the relation

Dedicated to G.G. Emch.

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among having a complete set of eigenvectors, the (lack of) commutativity, and compatibility. In Sect. 7, we prove that these pulled-back operators satisfy the physical axioms for a C* algebra and prove the informational completeness of them.

2 The C* Axioms for a Physical System

In 1947, Irving Segal [26] gave axioms for a physical system to form a groundwork for a C* algebra. This was mostly overlooked by people at the time. It was not until 1972 that G.G. Emch [11] gave a variation of the axioms that were accepted by many. We will give them here along with some examples that will be used later:

Axiom 1 For each physical system, Σ, we can associate the triple $(\mathfrak{A}, \mathfrak{S}, \langle ; \rangle)$ formed by the set $\mathfrak{A}$ of all its observables, the set $\mathfrak{S}$ of all its states, and a mapping $\langle ; \rangle : (\mathfrak{S}, \mathfrak{A}) \to \mathbb{R}$ which associates with each pair (ϕ, A) in $(\mathfrak{S}, \mathfrak{A})$ a real number $\langle \phi; A \rangle$ that we interpret as the expectation value of the observable A when the system is in the state ϕ.

Example Let $\mathcal{H}$ be a separable Hilbert space with inner product denoted. $\langle \cdot, \cdot \rangle$. Let A be a self-adjoint bounded operator on $\mathcal{H}$ (an observable) and let ρ be a self-adjoint bounded operator of trace class with trace 1 (a state). Then $\langle \rho; A \rangle = \mathrm{tr}(\rho A)$. If ρ is a vector state, then ρ is a projection, $\rho = P_\psi$, onto some normalized state vector $\psi \in \mathcal{H}$, and $\langle P_\psi; A \rangle = \langle \psi, A\psi \rangle$.

Definition 1 For a fixed $A \in \mathfrak{A}$, we have $\langle \cdot; A \rangle : \mathfrak{S} \to \mathbb{R}$. If $\mathcal{T} \subseteq \mathfrak{S}$, denote by $A|_\mathcal{T}$ the restriction $\langle \cdot; A \rangle : \mathcal{T} \to \mathbb{R}$. Declare $A|_\mathcal{T} \le B|_\mathcal{T}$ whenever $\langle \phi; A \rangle \le \langle \phi; B \rangle\ \forall \phi \in \mathcal{T}$. If $\mathcal{T} = \mathfrak{S}$, then we simply write $A \le B$. A subset $\mathcal{T}$ is said to be **full** with respect to a subset $\mathfrak{B} \subseteq \mathfrak{A}$ iff A and B in $\mathfrak{B}$, and $A|_\mathcal{T} \le B|_\mathcal{T} \Rightarrow A \le B$.

Example Let $\mathcal{H}$ be a separable Hilbert space. Let $\{\psi_i\}$ be an orthonormal basis for $\mathcal{H}$. For $\psi \in \mathcal{H}$ and $\|\psi\| = 1$, let $P_\psi \varphi = \langle \psi, \varphi \rangle \psi$. Let $\mathfrak{B}_\Psi = \{\sum_i \alpha_i P_{\psi_i}, \alpha_i \in \mathbb{C}, \sum_i |\alpha_i| < \infty\}$. Then $\mathcal{T}_\Psi = \{P_{\psi_i}\}$ is full with respect to $\mathfrak{B}_\Psi$.

Axiom 2 The relation $\le$ is a partial ordering relation on $\mathfrak{A}$.

Axiom 3 (i) There exist in $\mathfrak{A}$ two elements 0 and 1 such that, for all $\phi \in \mathfrak{S}$, we have $\langle \phi; 0 \rangle = 0$ and $\langle \phi; 1 \rangle = 1$.

(ii) For each observable $A \in \mathfrak{A}$ and any $\lambda \in \mathbb{R}$ there exists $(\lambda A) \in \mathfrak{A}$ such that $\langle \phi; \lambda A \rangle = \lambda \langle \phi; A \rangle$ for all $\phi \in \mathfrak{S}$.

(iii) For any pair of observables A and B in $\mathfrak{A}$ there exists an element $(A + B)$ in $\mathfrak{A}$ such that $\langle \phi; A + B \rangle = \langle \phi; A \rangle + \langle \phi; B \rangle$ for all $\phi \in \mathfrak{S}$.

Definition 2 Denote the set of all dispersion-free states for the observable A by $\mathfrak{S}_A$.

Remark At this stage we only have to define "a dispersion-free state for A"; i.e., a state, ϕ, for which, whenever we experimentally measure the observable A, we always obtain the same value. This value will automatically be the expectation $\langle \phi; A \rangle$. Defining the "dispersion" is more challenging, and we will not do it here. It may be defined in terms of "the quantum variance", or "the full width at half height", or "the bulk width (gross width)", etc. See [21, pp. 271–279] and references therein.

Definition 3 A subset $\mathcal{T} \subseteq \mathfrak{S}$ is said to be *complete* if it is full with respect to the subset $\mathfrak{A}_{\mathcal{T}} \subseteq \mathfrak{A}$ defined by $\mathfrak{A}_{\mathcal{T}} = \{A \in \mathfrak{A} \mid \mathfrak{S}_A \supseteq \mathcal{T}\}$. A complete subset $\mathcal{T} \subseteq \mathfrak{S}$ is said to be *deterministic* for a subset $\mathfrak{B} \subseteq \mathfrak{A}$ whenever $\mathfrak{B} \subseteq \mathfrak{A}_{\mathcal{T}}$. A subset $\mathfrak{B} \subseteq \mathfrak{A}$ is said to be *compatible* if the set $\mathfrak{S}_{\mathfrak{B}} \equiv \bigcap_{B \in \mathfrak{B}} \mathfrak{S}_B$ is complete.

Example $\mathcal{T}_\Psi$ is complete because it is full with respect to $\mathfrak{A}_{\mathcal{T}} \equiv \mathfrak{B}_\Psi$. It is moreover deterministic for any subset $\mathfrak{C} \subseteq \mathfrak{A}_{\mathcal{T}} \equiv \mathfrak{B}_\Psi$.

Example Compatibility of $\mathfrak{B}$ in $\mathcal{H}$ is known to be given by $AB = BA\ \forall A, B \in \mathfrak{B}$.

Axiom 4 The set $\mathfrak{S}_A$ is deterministic for the one-dimensional subspace of $\mathfrak{A}$ generated by A; for any two observables A and B we have $\mathfrak{S}_{A+B} \supseteq \mathfrak{S}_A \cap \mathfrak{S}_B$, and $\mathfrak{S}_1 = \mathfrak{S}$.

Example If in $\mathcal{H}$ we take a bounded observable A that has no eigenvalues, then $\mathfrak{S}_A$ is empty and Axiom 4 is unachievable. Several specific examples of this will be given at the end of this section.

Axiom 5 For any element A in the set of observables $\mathfrak{A}$ and any non-negative integer n, there is at least one element, denoted A^n, in $\mathfrak{A}$ such that (i) the set of dispersion-free states for A^n is contained in the set of dispersion-free states for A, (ii) $\langle \phi; A^n \rangle = \langle \phi; A \rangle^n$ for all ϕ in the set of dispersion-free states for A.

Definition 4 Let A and $B \in \mathfrak{A}$. $A \circ B$ is defined by $A \circ B \equiv \frac{1}{2}([A+B]^2 - A^2 - B^2)$.

Axiom 6 For any three observables A, B, and C in which A and C are compatible, $(A \circ B) \circ C - A \circ (B \circ C)$ vanishes.

Example In any Hilbert space setting, this axiom is automatically satisfied.

Axiom 7 The norm of $A \in \mathfrak{A}$, $\|A\| \equiv \sup_{\phi \in \mathfrak{S}} |\langle \phi; A \rangle|$, is finite and $\mathfrak{A}$ is topologically complete when regarded as a metric space with the distance between any two elements A and B of $\mathfrak{A}$ defined by $\|A - B\|$. $\mathfrak{S}$ is then identified with the set of all continuous positive linear functionals ϕ on $\mathfrak{A}$ satisfying $\langle \phi; 1 \rangle = 1$.

Axiom 8 A sufficient condition for a set $\mathfrak{B}$ of observables to be compatible is that $\overline{\mathfrak{P}(\mathfrak{B})}$ is associative. Here $\mathfrak{P}(\mathfrak{B})$ is the set of polynomials in $\mathfrak{B}$.

Axiom 9 $\mathfrak{A}$ can be identified with the set of all self-adjoint elements of a real or complex, associative, and involutive algebra $\mathfrak{R}$ satisfying

(i) For each $R \in \mathfrak{R}$ there exists an element A in $\mathfrak{A}$ such tht $R^*R = A^2$.
(ii) $R^*R = 0$ implies $R = 0$.

We mention Axiom 10 for completeness only. It is not necessary to obtain a C* algebra.

Axiom 10 To each pair of observables A and B in $\mathfrak{A}$ corresponds an observable C in $\mathfrak{A}$ in the sense that for all $\phi \in \mathfrak{S}$, we have

$$\langle \phi; (A - \langle \phi; A \rangle 1)^2 \rangle \langle \phi; (B - \langle \phi; B \rangle 1)^2 \rangle \geq \langle \phi; C \rangle^2.$$

All axioms except 4 and 5 hold in any Hilbert space with $\langle\phi; A\rangle = \mathrm{Tr}(\phi A)$ and $\mathfrak{A}$ consisting of bounded operators. To have axioms 4 and 5 satisfied as well, we will have to have $\mathfrak{A}$ consisting of (some) bounded operators with purely discrete spectrum.

A counter-example is provided by the position operator, Q, or the momentum operator, P. They are unbounded self-adjoint operators with a purely continuous spectrum on any of the non-relativistic Hilbert spaces in quantum mechanics. We first treat them as being generated by their spectral projections onto compact sets of their spectrum, each being a bounded self-adjoint operator. But you have a problem; each of these has a piecewise continuous spectrum, and thus has no purely discrete spectrum (eigenvalues) associated with it, much less a complete basis of eigenvectors. The same problem appears for any self-adjoint operator with a purely continuous spectrum in any Hilbert space. Next, we may use a theorem [28] that says that if T is a self-adjoint, bounded operator on a separable Hilbert space, then there exists a self-adjoint compact operator K such that $T + K$ has eigenvectors that span the space. Moreover, by a theorem of von Neumann [27], $\|K\|$ may be made arbitrarily small and T does not have to be bounded. But there is no physical interpretation for what K may be! We conclude that considering just Q or just P will not do for obtaining C* algebra from the physical perspective.

3 Informational Completeness

In any Hilbert space, we have

Definition 5 [20] A set of bounded self-adjoint operators on Hilbert space $\mathcal{H}$, $\{A_\beta \mid \beta \in I,\ I \text{ some index set}\}$, is *informationally complete* iff for all states ρ, ρ' on $\mathcal{H}$ such that $\mathrm{Tr}(\rho A_\beta) = \mathrm{Tr}(\rho' A_\beta)$ for all $\beta \in I$ then $\rho = \rho'$.

Example [20] In spinless quantum mechanics, the set of all spectral projections for position is not informationally complete. Neither is the set of all spectral projections for momentum, nor even the union of them. This is another reason why we will not base our theory on Q or on P.

There are two properties of an informationally complete set which we will note:

(1) If I is an informationally complete set in $\mathcal{H}$, then any bounded operator may be written as (the closure in the topology induced by the trace) an integral(s) over the set I [7].
(2) If $\dim(\mathcal{H})\rangle 1$, then no set of self-adjoint operators on $\mathcal{H}$ is informationally complete if it is a commuting set [8].

Thus, we shall look for a set that generates, somehow, all of $B(\mathcal{H})$ and necessarily is not a commuting set.

4 Phase Space

We shall only briefly summarize this section, as it has been discussed at the last meeting in this series [6].

Start with any dynamical group G such as the Galilei group or the Poincaré group. These are all Lie groups. Form the Lie algebra $\mathfrak{g}$, and then take its dual $\mathfrak{g}^*$. From the structure constants in the Lie algebra, construct the coboundary operator δ between the various $\wedge^n(\mathfrak{g}^*)$. Take $\omega \in Z^2(\mathfrak{g}) \equiv \{\varpi \in (\mathfrak{g} \wedge \mathfrak{g})^* \mid \delta(\varpi) = 0\}$. Define the sub-Lie algebra

$h_\omega \equiv \{\xi \in \mathfrak{g} \mid \omega(\xi, \cdot) = 0\}$. Exponentiate h_ω to obtain the subgroup H_ω. Then, assuming that H_ω is closed, form $\Gamma \equiv G/H_\omega$. A theorem [13, 21] says that (1) Γ is a transitive symplectic manifold (i.e., a phase space) with 2-form (essentially the Poisson bracket) being the pull-back of ω; (2) Γ has even dimension 2m and the m-th exterior product of this 2-form is the left-invariant measure μ; (3) if X is any symplectic space under G, then $X =$ a union of the Γs. As Γ is a phase space, obtain the canonical variables.

Now form the separable Hilbert space $L^2_\mu(\Gamma)$, and on $L^2_\mu(\Gamma)$ define the action of the group G by

$$[V^\alpha(g)\Psi](x) = \alpha(h(g^{-1}, x))\Psi(g^{-1}x),$$

where $\Psi \in L^2_\mu(\Gamma)$, h is a generalized co-cycle and α is a one-dimensional representation of H_ω. These representations up to a phase are important in representing the spin plus angular momentum correctly. The representation V^α is highly reducible.

Define the multiplication by measurable functions f of the phase space coordinates by

$$[A(f)\Psi](x) \equiv f(x)\Psi(x).$$

We point out that the $A(f)$s are a commuting set (including $f \in$ the set of canonical variables), and thus are not informationally complete in $L^2_\mu(\Gamma)$. However, they are physically identifiable as they are just multiplication by the fs which in turn are just functions of the phase space coordinates. $L^2_\mu(\Gamma)$ is not a quantum mechanical Hilbert space either, as it is not irreducible.

5 Quantum Mechanical Hilbert Spaces and Intertwining with $L^2_\mu(\Gamma)$

We obtain the quantum mechanical Hilbert spaces by the "Mackey machine" [17, 18] as the irreducible unitary representation spaces H with U the representation. These are the usual representation spaces encountered in any textbook on quantum mechanics.

Take any Borel section $\sigma : \Gamma = G/H_\omega \to G$, and then define "$\eta$ is admissible with respect to the section σ" iff

$$\int_\Gamma |\langle U(\sigma(x))\eta, \eta\rangle|^2 d\mu(x) < \infty.$$

Furthermore, if η is admissible with respect to σ, then we say that η is "α-admissible with respect to σ" iff $U(h)\eta = \alpha(h)\eta$ for all $h \in H_\omega$ where α is a one-dimensional representation of H_ω. We have shown that the set of "α-admissible vectors with respect to σ" is never empty for any of the usual representation spaces for quantum mechanics [5, 21].

Now define a map $W^\eta : \mathcal{H} \times \Gamma \to \mathbb{C}$ by

$$[W^\eta(\varphi)](x) \equiv \langle U(\sigma(x))\eta, \varphi\rangle.$$

We have the remarkable theorem that (1) W^η is a (linear) map from $\mathcal{H}$ to $L^2_\mu(\Gamma)$ whenever η is α-admissible. Furthermore, (2) W^η interwines:

$$W^\eta U(g) = V^\alpha(g)W^\eta,$$

(3) $W^\eta(\mathcal{H})$ is a closed subspace of $L^2_\mu(\Gamma)$, and so we may define the canonical projection

$$P^\eta : L^2_\mu(\Gamma) \to W^\eta(\mathcal{H}).$$

Next, we pull back the multiplication operators $A(f)$ from $L^2_\mu(\Gamma)$ to $\mathcal{H}$:

$$A^\eta(f) \equiv [W^\eta]^{-1} P^\eta A(f) W^\eta,$$

which is an operator acting from $\mathcal{H}$ to $\mathcal{H}$. Then for $\|\eta\| = 1$, we find that

$$A^\eta(f) \equiv \int_\Gamma f(x) T^\eta(x) d\mu(x),$$

$$T^\eta(f) \equiv P_{U(\sigma(x))\eta} = |U(\sigma(x))\eta\rangle\langle U(\sigma(x))\eta|,$$

i.e., there is an operator-valued density for $A^\eta(f)$. Since

$$U(g)T^\eta(x)U(g)^{-1} = T^\eta(gx),$$

and using the left-invariance of μ with respect to G, we have

$$\begin{aligned} U(g)A^\eta(f)U(g)^{-1} &= \int_\Gamma f(x) T^\eta(gx) d\mu(x) \\ &= \int_\Gamma f(g^{-1}y) T^\eta(y) d\mu(y) \\ &= A^\eta(g.f) \end{aligned}$$

where

$$[g.f](x) = f(g^{-1}x).$$

In other words, the A^η are covariant under the action of G. Moreover, take a vector ψ in $\mathcal{H}$ with $\|\psi\| = 1$, take P_ψ to be the one-dimensional projection onto the ray determined by ψ, and form

$$\begin{aligned} \mathrm{Tr}(P_\psi A^\eta(f)) &= \int_\Gamma f(x)\, \mathrm{Tr}(P_\psi T^\eta(x)) d\mu(x) \\ &= \int_\Gamma f(x) |\langle U(\sigma(x))\eta, \psi\rangle|^2 d\mu(x). \end{aligned}$$

This is the transition probability between $U(\sigma(x))\eta$ and ψ integrated over f, and provides an interpretation of the meaning of η which is directly related to the instrument with which you measure P_ψ. See [6] for more on the physical interpretability of η.

If we take the fs corresponding to canonical variables, we obtain $A^\eta(f) = Q_i$, P_i, or S_i and these have the correct (anti-)commutation relations [21]. (These fs are not compactly supported, however.) This is a direct consequence of having P^η in the expression for $A^\eta(f)$. In fact, it is known that the operators $A^\eta(f)$ include all polynomials in the operators p and q, for example.

The set

$$E \equiv \{A^\eta(f) \mid f \in L^1_\mu(\Gamma) \cap L^\infty_\mu(\Gamma),\ 0 \le f(x) \le 1\}$$

has been studied [24, 25]. E forms an effect algebra which is also an M.V. algebra, a Heyting algebra, and contains no non-trivial projections. It includes all the $A^\eta(\chi_C)$ where χ_C is the characteristic function for C, C a (compact) Borel set in Γ, $\mu(C) < \infty$. Thus E contains the phase space localization operators. Generally, E is the set of phase space fuzzy localization operators.

What we have to discuss is whether E, suitably generalized, is informationally complete. We have that E contains non-commuting operators, because of the presence of the P^η in the $A^\eta(f)$s. The operators in $\{A^\eta(f)\}$ are known to be informationally complete for a number of cases: massive spin-zero [1] and mass zero, arbitrary helicity [4] representations of the Poincaré group, the affine group [14], the Heisenberg group [14], and massive representations of the inhomogeneous Galilei group [2, 3, 21]. In all these cases, there was an extra condition on η guaranteeing informational completeness:

$$\langle U(g)\eta, \eta\rangle \neq 0 \quad \text{a.e. } g \in G.$$

6 Spectral Properties of the $A^\eta(f)$

We will prove that the $A^\eta(f)$ for $f \in L^1_\mu(\Gamma) \cap L^p_\mu(\Gamma)$, $p > 1$, have a purely discrete spectrum.

Definition 6 Let $\mathcal{H}$ be a Hilbert space, and let B be a compact operator on $\mathcal{H}$. Let $\{\beta_k\}$ denote the set of singular values (eigenvalues) of B. The *nth trace class*, B_n, is defined to be the set of all compact operators such that $\sum_k |\beta_k|^n < \infty$. In B_n we denote the norm by

$$\|B\|_{B_n} \equiv \left[\sum_k |\beta_k|^n\right]^{1/n}.$$

Then, by interpolation theory [19, 23], we have the

Theorem 1 *Let (X, Σ, μ) be a measure space, $\mathcal{H}$ be a Hilbert space, and $B : \Sigma \to \mathcal{B}(\mathcal{H})$ be a positive operator valued measure. If B has an operator density T such that $\|T_x\| \leq c$ $\forall x \in X$, and $\mathrm{Tr}(T_x) \leq k$ $\forall x \in X$, c and k constants, let $f \in L^p_\mu(X)$. Then $B(f) \equiv \int_X f(x) T_x d\mu(x)$ is a compact, bounded operator with $\|B(f)\| \leq c^{1/p}\|f\|_p$ and $\|B(f)\|_{B_p} \leq r(p)\|f\|_p$ for some constant $r(p)$. In the case $p = 1$, $r(p) = k$.*

Now, consider any Hilbert space for which the phase space formalism applies. We have that $T^\eta(x)$ is a one dimensional projection; so, $c = k = 1$. Also $X = \Gamma$ for $B(f) = A^\eta(f)$. Thus we have that the $A^\eta(f)$s are all compact for suitable fs.

In the $A^\eta(f)$, we wish to include $f = \chi_B$ for B any compact Borel set. For these, $\chi_B \in L^1_\mu(\Gamma) \cap L^\infty_\mu(\Gamma)$; so, we will take $f \in L^1_\mu(\Gamma) \cap L^\infty_\mu(\Gamma)$ from here on.

Using the property that the $A^\eta(f)$s are compact, we may use the result of P.A.M. Dirac [10] to get that two $A^\eta(f)$s commute iff there exists a complete set of simultaneous eigenstates of them. We then may deduce easily that they are functions of each other, without using the spectral theorem.

In addition to the set E defined above, we will also define

$$F \equiv \{A^\eta(f) \mid f \in L^1_\mu(\Gamma) \cap L^\infty_\mu(\Gamma),\ f \text{ real valued}\},$$

$$E^+ \equiv \{A^\eta(f) \mid f \in L^1_\mu(\Gamma) \cap L^\infty_\mu(\Gamma),\ 0 \leq f(x)\},$$

$$R \equiv \{A^\eta(f) \mid f \in L^1_\mu(\Gamma) \cap L^\infty_\mu(\Gamma),\ f \text{ complex valued}\}.$$

By the theorem, each of these consists of compact operators and may be represented by

$$A^\eta(f) = \sum_i \lambda_i P_{\psi_i}$$

for some orthonormal basis $\{\psi_i\}$ of $\mathcal{H}$. Consequently, we may generate a C* algebra (for a physical system) from any of them and henceforth call the C* algebra "$\mathfrak{A}$". (One must be aware that in Axiom 3 we have $A^\eta(f) + A^\eta(h) = A^\eta(f+h)$, where all three of $A^\eta(f)$, $A^\eta(h)$, and $A^\eta(f+h)$ may have distinctly different eigenvectors!)

These $A^\eta(f) = \sum_i \lambda_i P_{\psi_i}$ are interesting, as they have a complete set of orthonormal eigenvectors despite f involving all the canonical variables. They are bounded. Different $A^\eta(f)$s generally do not commute. They are all compatible. Thus, commutability and compatibility are independent. This is a partial generalization of [9] in which it was pointed out that P.O.V.M.s may be compatible without commuting.

To measure the $A^\eta(f)$s in state ρ, we must first measure a single $A^\eta(\chi_{B_1})$ which is a countable process, then measure another one, and continue with $\{B_i \mid i \in \mathbb{N}\}$ constituting a subset of the Borel sets that accumulate at all rational points in Γ. From this countable process, all operators are in fact determined (once we prove the informational completeness).

7 Representations of the C* Algebra

We will follow the G.N.S. construction [12, 26] to obtain a representation of the C* algebra acting on some Hilbert space, and then show that this Hilbert space is equivalent to the original Hilbert space $\mathcal{H}$.

First we choose any state ϕ in the original Hilbert space $\mathcal{H}$ in which the localization operators, $A^\eta(f)$, were defined, and form $\langle \phi; A\rangle$, $A \in \mathfrak{A}$. Then let

$$\mathfrak{K}_\phi = \{K \in \mathfrak{A} \mid \langle \phi; R^* K\rangle = 0 \ \forall R \in \mathfrak{A}\},$$

which by the Cauchy–Schwarz–Buniakowski inequality is equal to

$$\mathfrak{K}_\phi = \{K \in \mathfrak{A} \mid \langle \phi; K^* K\rangle = 0\}.$$

(Note: This includes $\phi = P_\psi$ for any $\psi \in \mathcal{H}$.)

Since $\mathcal{H}$ is irreducible, any vector in $\mathcal{H}$ is cyclic; so, we will take $\phi = P_\psi$, with ψ of the form $U(g^{-1})\eta$, $g \in G$. Then in particular, abusing the notation for $\mathfrak{K}_\phi$, we have

$$\begin{aligned}\mathfrak{K}_{U(g^{-1})\eta} &= \{A^\eta(f) \in \mathfrak{A} \mid \|A^\eta(f)U(g^{-1})\eta\| = 0\}\\ &= \{A^\eta(f) \in \mathfrak{A} \mid \|A^\eta(g.f)\eta\| = 0\}.\end{aligned}$$

Since the set of fs is invariant under the group G, it suffices to consider just

$$\begin{aligned}\mathfrak{K}_\eta &= \{A^\eta(f) \in \mathfrak{A} \mid \|A^\eta(f)\eta\| = 0\}\\ &= \{A^\eta(f) \in \mathfrak{A} \mid A^\eta(f)\eta = 0\}.\end{aligned}$$

Now

$$A^\eta(f)\eta = \int f(x)\langle U(\sigma(x))\eta, \eta\rangle U(\sigma(x))\eta d\mu(x).$$

If $\langle U(\sigma(x))\eta, \eta\rangle = 0$ a.e. x for $\sigma(x)$ in some compact set $\mathcal{O}$ with non-empty interior, then for all f with support in $\mathcal{O}$, $A^\eta(f)\eta = 0$. Thus $\mathfrak{K}_\eta \neq \{0\}$ in a way that is invariant under all infinitesimal transformations. If $\langle U(\sigma(x))\eta, \eta\rangle \neq 0$ a.e. $\sigma(x)$ with $x \in \Gamma$, there may be some fs such that $A^\eta(f)\eta = 0$, but $A^\eta(g.f)\eta = 0$ does not hold for all g infinitesimally in all directions. Thus, $A^\eta(f)\eta = 0$ holds only for a thin set of fs.

Remarks: (1) If $A^\eta(f) \in E$, resp. $A^\eta(f) \in -E$, then since $\operatorname{spec}(A^\eta(f)) \subseteq (0, 1)$, resp. $\subseteq (-1, 0)$, $A^\eta(f)\eta \neq 0$.

(2) The α-admissibility of η implies

$$\begin{aligned} &\langle U(\sigma(x))\eta, \eta\rangle \neq 0 \quad \text{a.e. } \sigma(x) \text{ with } x \in \Gamma \\ \iff \quad &\langle U(g)\eta, \eta\rangle \neq 0 \quad \text{a.e. } g \in G; \end{aligned}$$

i.e., the same condition for obtaining informational completeness in the previously known cases for the Galilei and Poincaré groups.

(3) By using the modular function we obtain: if η is admissible, then $U(g)\eta$ is admissible for all $g \in G$. Moreover, we have: if η is α-admissible, then $U(h)\eta$ is α-admissible for all $h \in H$. Thus, coupled with the results above, we have that any vector of the form $U(h)\eta$, $h \in H$, will be suitable for obtaining the results below.

We will now obtain a representation of $\mathfrak{A}/\mathfrak{K}_\eta$, η satisfying $\langle U(g)\eta, \eta\rangle \neq 0$. For R, S in $\mathfrak{A}/\mathfrak{K}_\eta$, define $(R, S) \equiv \langle P_\eta; R^*S\rangle$. This turns out to be a sesquilinear form and generates a norm on $\mathfrak{A}/\mathfrak{K}_\eta$. Hence $\mathfrak{A}/\mathfrak{K}_\eta$ is a pre-Hilbert space which has the Hilbert space $\mathcal{H}_\eta$ as its completion. Note that we are taking the completion in the topology dual to the strong or weak sense, by the cyclicity of η. This is the same topology as the topology for which informational completeness is discussed in [21].

The representation π_η of $\mathfrak{A}$ is defined by

$$\begin{aligned} &\pi_\eta(R) : \mathfrak{A}/\mathfrak{K}_\eta \to \mathfrak{A}/\mathfrak{K}_\eta, \\ &\pi_\eta(R)S = RS. \end{aligned}$$

The G.N.S. theorem then proceeds to show that $\pi_\eta(R)$ can be extended to a bounded operator on $\mathcal{H}_\eta$. Taking $\mathfrak{A}/\mathfrak{K}_\eta$ instead of $\mathfrak{A}$ is moot when we operate on $\mathcal{H}_\eta$. The algebra generated by the set $\{A^\eta(f)\}$ is informationally complete in this representation.

Now for all practical purposes, we have that $\mathcal{H}_\eta \subseteq \mathcal{H}$. Define

$$\mathcal{U}(g) : A^\eta(f) \mapsto A^\eta(g.f), \quad g \in G.$$

$\mathcal{U}$ is a representation of G on $\mathfrak{A}$. But using the covariance property of the $A^\eta(f)$, we see that this representation of the symmetry group is given by $\mathcal{U}(g)A^\eta(f) = U(g)A^\eta(f)U^{-1}(g)$; i.e. by the same U we had before. But that U is irreducible, and hence the Hilbert space obtained through the G.N.S. construction is the "same" as the original Hilbert space and $\pi_\eta(\mathfrak{A})\tilde{} = B(\mathcal{H})$. Consequently,

Theorem 2 *The algebra generated by the set* $\{A^\eta(f) \mid f \in L^1_\mu(\Gamma) \cap L^\infty_\mu(\Gamma), f$ *real-valued*, η α*-admissible in* $\mathcal{H}$ *and* $\langle U(g)\eta, \eta\rangle \neq 0$ a.e. $g \in G\}$ *is informationally complete in the* G*-irreducible representation space* $\mathcal{H}$*, for any* G *that is a Lie group and* Γ *is a phase space coming from* G *via the coboundary method.*

8 Conclusion

For certain $\eta \in \mathcal{H}$, we have exhibited a set $\{A^\eta(f) \mid f$ real valued, $f \in L^1_\mu(\Gamma) \cap L^\infty_\mu(\Gamma)\}$ of operators that have a physical meaning in any experiment in which one measures by quantum mechanical means. These $A^\eta(f)$ each have a full set of eigenvectors. They are compatible and are generally non-commuting. They form a C* algebra, and hence form a foundation for the C*-algebraic formalism for physics in the free case. We may use the G.N.S. construction to obtain the informational completeness of the algebra generated by the set $\{A^\eta(f) \mid f$ real valued, $f \in L^1_\mu(\Gamma) \cap L^\infty_\mu(X)\}$. Generalizing to any physical system that has the phase space localization operators on it, we obtain a C* algebra and the informational completeness of the set generated from these $A^\eta(f)$.

Those people interested in quantum field theory may be interested in how the phase space formalism extends to that realm. See [15, 16] for dealing with simple systems such as the ones exhibiting Heisenberg symmetry and [22] for general systems.

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